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AN ASYMPTOTIC SOLUTION OF
LINEAR SECOND-ORDER
HYPERBOLIC DIFFERENTIAL EQUATIONS

by

MORRIS KLINE

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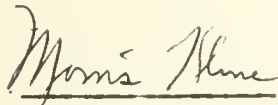
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Project Director

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ABSTRACT

This paper presents a method of obtaining one type of asymptotic solution for the linear hyperbolic second-order partial differential equation in $n-1$ independent variables and one time variable with coefficients independent of time. Appropriate initial and boundary conditions are specified. In particular it is supposed that a time-harmonic forcing term is present either in the non-homogeneous term of the differential equation or in a boundary condition. Then the "steady state" part of the exact solution has the same time behavior. The spatial behavior of the "steady state" part is then expressed asymptotically as a series in positive integral powers of $\frac{1}{\omega}$, where ω is the circular frequency of the source. The successive coefficients of this series are respectively discontinuities of a particular solution of the differential equation and of successive time-derivatives of this solution.

By a study of discontinuities of solutions of the differential equation it is shown that the discontinuities needed for the successive coefficients of the expansion can be obtained as solutions of a recursive system of first-order linear ordinary differential equations. Initial conditions to fix the solutions of this recursive system can be specified in some cases.

I. Introduction

In view of the difficulty in obtaining exact solutions for problems in partial differential equations with prescribed initial and boundary conditions, attention has been given in the literature to solutions in the form of asymptotic series. The present paper presents a method for obtaining asymptotic solutions for certain classes of problems involving linear second-order hyperbolic differential equations. The problems considered arise directly in physical situations or result from reducing vectorial problems of electromagnetic theory to scalar problems.

Briefly stated, the contents of this paper is as follows. We consider the linear hyperbolic partial differential equation

$$(1.1) \quad L(u) = \sum_{i,j=1}^n a^{ij} u_{ij} + \sum_{k=1}^n b^k u_k + cu = 0,$$

wherein u is a function of $x_1, x_2, \dots, x_n$; the coefficients a^{ij} , b^k , and c are functions of $x_1, x_2, \dots, x_{n-1}$; and $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. Associated initial and boundary conditions are specified later. We need only note here that one of the boundary conditions is non-homogeneous. We also consider the case wherein a special type of non-homogeneous term, representing a source or applied force, appears on the right hand side. We shall write $u(x,t)$ for $u(x_1, x_2, \dots, x_{n-1}, x_n)$, letting x stand for the first $n-1$ variables and t for x_n .

Among solutions of equation (1.1) we consider a special solution, $U(x,t)$, called a pulse solution, which corresponds to a source or impressed force whose time behavior is the Heaviside unit function. We then introduce a Duhamel principle which expresses the solution $u(x,t)$ due to a source or impressed force with arbitrary time behavior in terms of an integral involving $U(x,t)$. If we now restrict the arbitrary time behavior to a harmonic one, namely $e^{-i\omega t}$, wherein ω is the circular frequency of the impressed force, it is possible to show first that with increased time $u(x,t)$ takes

the form $\bar{u}(x)e^{-i\omega t}$. The function $\bar{u}(x)$ may be described as the spatial part of the "steady state" behavior of $u(x,t)$. It is possible next by applying integration by parts to the Duhamel integral to obtain an asymptotic series for $\bar{u}(x)$. The series runs as follows:

$$(1.2) \quad \bar{u}(x) = \sum_{\alpha} [\bar{U}(x, r_{\alpha})] e^{i\omega r_{\alpha}} - \frac{1}{i\omega} \sum_{\alpha} [\bar{U}_t(x, r_{\alpha})] e^{i\omega r_{\alpha}} + \frac{1}{(i\omega)^2} \sum_{\alpha} [\bar{U}_{tt}(x, r_{\alpha})] e^{i\omega r_{\alpha}} - \dots$$

The essential variable of the series is $1/\omega$ and the successive terms contain increasing positive integral powers of $1/\omega$. The first term of the series is independent of ω and hence is the limiting value as $\omega \rightarrow \infty$. This term gives, therefore, the "geometrical optics" limit of the quantity $\bar{u}(x)$. The value of this first term at any point x in space, apart from the phase factor, is the sum of the discontinuities with respect to t in $U(x,t)$ at that same x . The essential part of the coefficient of the second term of the series is the sum of the discontinuities with respect to t in $U_t(x,t)$; and so on for the higher coefficients.

The very form of the asymptotic series tells us that the behavior of solutions of partial differential equations is intimately related to the discontinuities of the pulse solution. However the primary objective of this paper is to show how to obtain the coefficients of the expansion without knowing the pulse solution itself. For this purpose we study discontinuities of solutions of equation (1.1). To treat discontinuous solutions we replace the differential equation by a more general integral equation which admits discontinuous solutions, and derive conditions which the discontinuities in U , U_t , U_{tt} , etc., must satisfy.

There results first of all the fact that if such discontinuities lie on a surface $\phi(x_1, \dots, x_n) = 0$ then ϕ must satisfy the characteristic condition

$$(1.3) \quad \sum_{i,j=1}^n a^{ij} \phi_i \phi_j = 0 \quad \text{on } \phi = 0.$$

The most useful conditions which follow from the study of discontinuities are what may be called the transport and higher transport equations. If one follows a dis-

continuity in U , $[U]$, along a path on $\phi = 0$ which has at each point the direction called the transverse direction, i.e., $\dot{x}_i(s) = \sum_{k=1}^n a^{ik} \phi_k$, then along this path $[U]$ satisfies the transport equation

$$(1.4) \quad \left(\sum_{i,j=1}^n a^{ij} \phi_{ij} + \sum_{k=1}^n b^k \phi_k \right) [U] + 2 \frac{\partial [U]}{\partial s} = 0.$$

Likewise, a discontinuity in U_t satisfies the higher transport equation

$$(1.5) \quad \left(\sum_{i,j=1}^n a^{ij} \phi_{ij} + \sum_{k=1}^n b^k \phi_k \right) [U_t] + 2 \frac{\partial [U_t]}{\partial s} = - \sum_{i,j=1}^{n-1} a^{ij} \frac{\partial^2 [U]}{\partial x_i \partial x_j} - \sum_{k=1}^{n-1} b^k \frac{\partial [U]}{\partial x_k} - c[U].$$

Entirely analogous equations hold for the discontinuities in U_{tt} , etc. Solution of these transport equations would give the values of the discontinuities required in the asymptotic series (1.2). Initial conditions required to fix the solutions of these transport equations cannot be prescribed generally, but some indications of what can be used in special cases are given.

A word should be said about the range of validity of the asymptotic series of this paper. It should be expected that a series in positive integral powers of $\frac{1}{\omega}$ cannot be applicable everywhere. It cannot be expected to hold on the boundary of shadow regions nor in the neighborhood of foci and caustics. Other expansions involving fractional and negative powers of $\frac{1}{\omega}$ must obtain in such regions. Such expansions are being considered, and a forthcoming paper by Mr. Irvin Kay will contain some results on these expansions. Another point now being investigated is a mathematical criterion for the range of validity of the present expansion. We give here a criterion in terms of the properties of $U(x,t)$. That $U(x,t)$ possesses these properties may be known on physical grounds but it cannot be ascertained at present by examination of the differential equation and associated initial and boundary conditions.

The suggestion for this work derives from an analogous development carried out by the writer for solutions of Maxwell's equations¹ following ideas introduced by R.K. Luneberg. The present paper treats one second-order partial differential equation

1. See Kline, M.: Asymptotic Solution of Maxwell's Equations, Research Report No. EM-24 submitted by New York University to the Geophysics Research Division of the Air Force Cambridge Research Laboratories or Communications on Pure and Applied Mathematics, Vol. IV, 1951, pp. 225-262.

as opposed to the system of first-order Maxwell equations. While the general principles underlying this development are the same as those previously employed, the details of proof are quite different and the resulting ordinary differential equations for the coefficients of the expansion are also quite different in form. It is hoped that the present material will be easier to apply because the ordinary differential equations to be solved are simpler. For example, each ordinary differential equation here is the analogue of two sets of three ordinary differential equations in the electromagnetic case.

Aside from papers currently appearing on the asymptotic solution of ordinary differential equations and on various types of non-linear partial differential equations, several have appeared recently which bear directly on this paper or on the analogous electromagnetic case². The general background for the exposition in this paper will be found in Courant-Hilbert: Methoden der Mathematischen Physik (Springer, Berlin, 1937), V.II, Chapter 6, Sections 1 and 2 and Appendix, section 4.

2. See, for example, Friedlander, F.G.: Geometrical Optics and Maxwell's Equations, Proc. Cambridge Phil. Soc., V.43, Part 2, Apr. 1947, pp.284-286; Bremmer, H.: The Jumps of Discontinuous Solutions of the Wave Equation, Communications on Pure and Applied Mathematics, V.IV, 1951, pp. 419-426; Copson, E.T.: The Transport of Discontinuities in an Electromagnetic Field, Ibid, pp.427-433; Lowell, Sherman C.: The Propagation of Waves in Shallow Water, Ibid, V.II, 1949, pp. 275-291.

II. Formulation of the Problem

We shall be concerned with obtaining asymptotic expansions for functions $u(x_1, x_2, \dots, x_n)$ satisfying certain second-order partial differential equations with associated initial and boundary conditions. There are two types of problems to which the theory of asymptotic solutions would apply.

Problem I. Consider the homogeneous linear hyperbolic second-order partial differential equation

$$(2.1) \quad L(u) = \sum_{i,j=1}^n a^{ij} u_{ij} + \sum_{k=1}^n b^k u_k + cu = 0,$$

wherein u is a function of $x_1, x_2, \dots, x_n$, and the coefficients are functions of $x_1, x_2, \dots, x_{n-1}$. The function u is required to have all first and second-order derivatives in some domain of n -dimensional space. The coefficients a^{ij} are likewise required to possess all first and second-order derivatives in this domain.

Let S_1 be a hypersurface in the $(n-1)$ -dimensional space of the variables $x_1, x_2, \dots, x_{n-1}$. S_1 may be infinite in extent or closed. In the latter case we shall be interested in the solution u in the space on and exterior to S_1 . As initial conditions we require that

$$(2.1a) \quad u(x, t) = 0 \quad \text{and} \quad u_t(x, t) = 0 \quad \text{for } t < 0 \quad \text{and for all } x.$$

$$(2.1b) \quad u(x, 0) = 0 \quad \text{and} \quad u_t(x, 0) = 0 \quad \text{for } x \text{ off } S_1.$$

Our boundary condition is

$$(2.1c) \quad u(x, t) = h(x)f(t) \quad \text{for } x \text{ on } S_1.$$

It is assumed that $f(t) = 0$ for $t < 0$ and that $f(t)$ and $f'(t)$ are continuous.

For this problem I we define as the pulse solution $U(x, t)$ that solution which satisfies the conditions (2.1), (2.1a), (2.1b), and (2.1c) except that $f(t)$ in (2.1c) is the Heaviside unit function $\chi(t)$, which is 0 for $t < 0$ and 1 for $t > 0$. We require further of $U(x, t)$ that every bounded domain of (x, t) space be decomposable into a finite number of closed subdomains in each of which U, U_t, U_{tt} , and the higher time-derivatives exist and possess all first and second-order continuous derivatives. Moreover all discontinuities in U, U_t, U_{tt} , etc., with respect to t must be finite.

Problem II. Consider the non-homogeneous linear hyperbolic second-order partial differential equation

$$(2.2) \quad L(u) = h(x)f(t),$$

where $h(x)$ is not zero in some closed $(n-1)$ -dimensional region V of $x_1, \dots, x_{n-1}$ space, $f(t) = 0$ for $t < 0$, and $f(t)$ and $f'(t)$ are continuous for $t > 0$. The initial conditions are:

$$(2.2a) \quad u(x,t) = 0 \text{ and } u_t(x,t) = 0 \text{ for } t < 0 \text{ and for all } x.$$

$$(2.2b) \quad u(x,0) = 0 \text{ and } u_t(x,0) = 0 \text{ for } x \text{ outside of } V.$$

We may, if we wish, impose the following boundary condition: Let S_1 be any surface in $(n-1)$ -dimensional space:

$$(2.2c) \quad u(x,t) = 0 \text{ for } x \text{ on } S_1 \text{ and for all } t.$$

For problem II we also define as the pulse solution the function $U(x,t)$ which satisfies (2.2) except that $f(t)$ is the special function $\eta(t)$, and $U(x,t)$ satisfies the initial and boundary conditions (2.2a), (2.2b), and (2.2c). The additional conditions imposed on $U(x,t)$ in problem I as to the existence of derivatives in closed subdomains and as to the finiteness of the discontinuities in t are imposed here too.

Problem III. This problem is essentially a combination of the two preceding ones. We consider the non-homogeneous linear hyperbolic second-order partial differential equation

$$(2.3) \quad L(u) = h_1(x)f_1(t),$$

where $h_1(x) \neq 0$ in some closed $(n-1)$ -dimensional volume V of $x_1, \dots, x_{n-1}$ space bounded by the surface S , $f_1(t) = 0$ for $t < 0$, and $f(t)$ and $f'(t)$ continuous for $t > 0$. The initial conditions are

$$(2.3a) \quad u(x,t) = 0 \text{ and } u_t(x,0) = 0 \text{ for } t < 0 \text{ and for all } x;$$

$$(2.3b) \quad u(x,0) = 0 \text{ and } u_t(x,0) = 0 \text{ for } x \text{ outside of } V.$$

In addition, let S_1 be any closed surface on which we impose the boundary condition

$$(2.3c) \quad u(x,t) = h_2(x)f_2(t) \text{ for } x \text{ on } S_1,$$

where $f_2(t)$ and $f_2'(t)$ are continuous and $f_2(t) = 0$ for $t < 0$.

Problem III can be reduced to problems I and II. We have but to let $u = u_1 + u_2$, where

$$L(u_1) = h_1(x)f_1(t),$$

$$u_1(x,t) = \frac{\partial u_1}{\partial t}(x,t) = 0 \text{ for } t < 0 \text{ and for all } x,$$

$$u_1(x,0) = \frac{\partial u_1}{\partial t}(x,0) = 0 \text{ for } x \text{ outside of } V,$$

$$u_1(x,t) = 0 \text{ for } x \text{ on } S_1 \text{ and for all } t;$$

and

$$L(u_2) = 0,$$

$$u_2(x,t) = \frac{\partial u_2}{\partial t}(x,t) = 0 \text{ for } t < 0 \text{ and for all } x,$$

$$u_2(x,0) = \frac{\partial u_2}{\partial t}(x,0) = 0 \text{ for } x \text{ outside of } V,$$

$$u_2(x,t) = h_2(x)f_2(t) \text{ for } x \text{ on } S_1.$$

III. The Duhamel Principle

The pulse solution we have introduced for the several problems will generally be discontinuous at some values of x and t because it corresponds to a sudden disturbance introduced at $t = 0$ either on some boundary S_1 (problem I) or in some region V (problem II). This disturbance will propagate as a discontinuity which will reach point x off S_1 , or outside of V , at some later time t . We have required that in any bounded region of (x,t) space there be closed subdomains within which U and its derivatives to second order be continuous. However across the boundaries of such closed subdomains, U , U_t , U_{tt} , etc., may be discontinuous. We shall suppose that the discontinuities lie on some hypersurface $\phi(x_1, x_2, \dots, x_{n-1}, t) = 0$, that $\phi = 0$ may be solved explicitly for t in the neighborhood of a given value t_1 of t , and that the explicit solution is given by $t_1 = \psi(x_1, \dots, x_{n-1})$, where ψ possesses first- and second-order derivatives with respect to its variables.

Since U is discontinuous across the boundaries of the closed subdomains we shall require that certain discontinuity conditions, (3.11) and (3.12), hold on $\phi = 0$. Under these conditions and the general assumptions made about $U(x,t)$, the following assertion holds:

$$(3.1) \quad u(x,t) = \frac{\partial}{\partial t} \int_0^t U(x,t-\tau) f(\tau) d\tau.$$

This integral which relates the solutions $u(x,t)$ and $U(x,t)$ is known as Duhamel's integral³.

We shall verify that (3.1) is correct for problem I and then note any special considerations that may be needed to take care of problem II.

In verifying (3.1) we shall take into account a discontinuity at $t = t_1$ of $U(x,t)$. Let us suppose, then, that $U(x,t)$ is discontinuous at $t = t_1$ and that $0 < t_1 < t$. Since we have required that $U(x,t)$ be sectionally smooth and that it satisfy (2.1) in each closed domain in which it is sectionally smooth, we shall substitute (3.1) into (2.1). First, we note that from (3.1) we may write

$$(3.2) \quad \int_0^t u(x,s) ds = \int_0^t U(x,t-\tau) f(\tau) d\tau.$$

In (2.1) we write s for t and integrate with respect to s from 0 to t . Since the coefficients of (2.1) are assumed to be independent of t , we may write

3. Cf. Courant, R. and Hilbert, D: Methoden der Mathematischen Physik, V.II, p.184, Springer, Berlin, 1937.

$$(3.3) \sum_{i,j=1}^n a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t u(x,s) ds + \sum_{k=1}^n b^k \frac{\partial}{\partial x_k} \int_0^t u(x,s) ds + c \int_0^t u(x,s) ds = 0.$$

We may now use (3.2) to write

$$(3.4) \sum_{i,j=1}^n a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t U(x,t-\gamma) f(\gamma) d\gamma + \sum_{k=1}^n b^k \frac{\partial}{\partial x_k} \int_0^t U(x,t-\gamma) f(\gamma) d\gamma + c \int_0^t U(x,t-\gamma) f(\gamma) d\gamma = 0.$$

Since $U(x,t)$ is discontinuous at $t = t_1$, $U(x,t-\gamma)$ is discontinuous at $t-\gamma = t_1$ or $\gamma = t-t_1$. We therefore shall break up the γ -interval of integration into 0 to $t-t_1-$ and $t-t_1+$ to t , where by $t-t_1-$ we mean $t-t_1-\epsilon$ and by $t-t_1+$ we mean $t-t_1+\epsilon$. We shall use the fact that $t_1 = \psi(x_1, x_2, \dots, x_{n-1})$.

To verify that (3.4) is correct we shall consider the effect of the differentiations in each term of (3.4). Consider first a term such as

$$a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t U(x,t-\gamma) f(\gamma) d\gamma, \text{ for } i, j = 1, 2, \dots, n-1.$$

$$\begin{aligned} & a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t U(x,t-\gamma) f(\gamma) d\gamma \\ &= a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \int_0^{t-t_1-} U(x,t-\gamma) f(\gamma) d\gamma + a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \int_{t-t_1+}^t U(x,t-\gamma) f(\gamma) d\gamma \\ &= a^{ij} \frac{\partial}{\partial x_i} \left\{ \int_0^{t-t_1-} U_j(x,t-\gamma) f(\gamma) d\gamma - U(x,t_1+) f(t-t_1) \psi_j \right. \\ & \quad \left. + \int_{t-t_1+}^t U_j(x,t-\gamma) f(\gamma) d\gamma + U(x,t_1-) f(t-t_1) \psi_j \right\}. \end{aligned}$$

We now carry out the differentiation with respect to x_1 . In differentiating the second and fourth terms we note that we have a product of three functions in which product the first term is a compound function of x_1 . Hence

$$a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t U(x, t-\gamma) f(\gamma) d\gamma = a^{ij} \left\{ \int_0^{t-t_1-} U_{ij}(x, t-\gamma) f(\gamma) d\gamma - U_j(x, t_1+) f(t-t_1) \psi_i \right. \\ - U_i(x, t_1+) f(t-t_1) \psi_j - U_t(x, t_1+) f(t-t_1) \psi_i \psi_j + U(x, t_1+) f'(t-t_1) \psi_i \psi_j \\ - U(x, t_1+) f(t-t_1) \psi_{ij} + \int_{t-t_1+}^t U_{ij}(x, t-\gamma) f(\gamma) d\gamma + U_j(x, t_1-) f(t-t_1) \psi_i \\ + U_i(x, t_1-) f(t-t_1) \psi_j + U_t(x, t_1-) f(t-t_1) \psi_i \psi_j - U(x, t_1-) f'(t-t_1) \psi_i \psi_j \\ \left. + U(x, t_1-) f(t-t_1) \psi_{ij} \right\}.$$

We may shorten this result somewhat by using the notation $[U(x, t_1)]$ to mean $U(x, t_1+) - U(x, t_1-)$. Then, for $i, j = 1, 2, \dots, n-1$,

$$(3.5) \quad a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t U(x, t-\gamma) f(\gamma) d\gamma = a^{ij} \left\{ \int_0^t U_{ij}(x, t-\gamma) d\gamma + [U(x, t_1)] f'(t-t_1) \psi_i \psi_j \right. \\ - [U(x, t_1)] f(t-t_1) \psi_{ij} - [U_i(x, t_1)] f(t-t_1) \psi_j - [U_j(x, t_1)] f(t-t_1) \psi_i \\ \left. - [U_t(x, t_1)] f(t-t_1) \psi_i \psi_j \right\}.$$

We consider next terms of the first sum in (3.4) for which $j = n$ and i is $1, 2, \dots, n-1$.

$$a^{it} \frac{\partial^2}{\partial x_i \partial t} \int_0^t U(x, t-\gamma) f(\gamma) d\gamma = a^{it} \frac{\partial^2}{\partial x_i \partial t} \int_0^{t-t_1-} U(x, t-\gamma) f(\gamma) d\gamma + a^{it} \frac{\partial^2}{\partial x_i \partial t} \int_{t-t_1+}^t U(x, t-\gamma) f(\gamma) d\gamma, \\ = a^{it} \frac{\partial}{\partial x_i} \left\{ \int_0^{t-t_1-} U_t(x, t-\gamma) f(\gamma) d\gamma + U(x, t_1+) f(t-t_1) + \int_{t-t_1+}^t U_t(x, t-\gamma) f(\gamma) d\gamma \right. \\ \left. + U(x, 0) f(t) - U(x, t_1-) f(t-t_1) \right\}.$$

Before carrying out the next differentiation we shall drop the term containing $U(x, 0)$, since by the initial conditions this is 0 except when x is on boundary surface S_1 . The latter case will be considered separately.

Now

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$$\begin{aligned}
 a^{it} \frac{\partial^2}{\partial x_i \partial t} \int_0^t U(x, t-\tau) f(\tau) d\tau &= a^{it} \left\{ \int_0^{t-t_1-} U_{it}(x, t-\tau) f(\tau) d\tau - U_t(x, t_1+) f(t-t_1) \psi_i \right. \\
 &+ U_i(x, t_1+) f(t-t_1) + U_t(x, t_1+) f(t-t_1) \psi_i - U(x, t_1+) f'(t-t_1) \psi_i \\
 &+ \int_{t-t_1+}^t U_{it}(x, t-\tau) f(\tau) d\tau + U_t(x, t_1-) f(t-t_1) \psi_i - U_i(x, t_1-) f(t-t_1) \\
 &\left. - U_t(x, t_1-) f(t-t_1) \psi_i + U(x, t_1-) f'(t-t_1) \psi_i \right\}. \\
 (3.6) \quad a^{it} \frac{\partial^2}{\partial x_i \partial t} \int_0^t U(x, t-\tau) f(\tau) d\tau &= a^{it} \left\{ \int_0^t U_{it}(x, t-\tau) f(\tau) d\tau - [U(x, t_1)] f'(t-t_1) \psi_i \right. \\
 &\left. + [U_i(x, t_1)] f(t-t_1) \right\}.
 \end{aligned}$$

Those second-order terms of (3.4) for which $i = n$ and $j = 1, 2, \dots, n-1$ give the same result as (3.6), since $a^{ij} = a^{ji}$, and the order of the differentiation is immaterial⁴.

We now consider the second-order term in (3.4) for which $i = j = n$. The first of the differentiations to be performed in this connection was done in deriving (3.6).

Hence

$$\begin{aligned}
 a^{tt} \frac{\partial^2}{\partial t^2} \int_0^t U(x, t-\tau) f(\tau) d\tau &= a^{tt} \frac{\partial}{\partial t} \left\{ \int_0^{t-t_1-} U_t(x, t-\tau) f(\tau) d\tau + U(x, t_1+) f(t-t_1) \right. \\
 &\left. + \int_{t-t_1+}^t U_t(x, t-\tau) f(\tau) d\tau - U(x, t_1-) f(t-t_1) \right\}, \\
 &= a^{tt} \left\{ \int_0^{t-t_1-} U_{tt}(x, t-\tau) f(\tau) d\tau + U_t(x, t_1+) f(t-t_1) + U(x, t_1+) f'(t-t_1) \right. \\
 &\left. + \int_{t-t_1+}^t U_{tt}(x, t-\tau) f(\tau) d\tau + U_t(x, 0) f(t) - U_t(x, t_1-) f(t-t_1) - U(x, t_1-) f'(t-t_1) \right\}.
 \end{aligned}$$

We use the initial condition that $U_t(x, 0) = 0$ for x off the boundary surface S_1 , and obtain

4. In performing the differentiation with respect to x_j first and then with respect to t one does end up with the term $U_j(x, 0) f(t)$ in addition to the other terms in (3.6). However since $U(x, t) = 0$ for $t = 0$ except on S_1 , we conclude that $U_j(x, 0) = 0$.

$$(3.7) \quad a^{tt} \frac{\partial^2}{\partial t^2} \int_0^t U(x, t-\gamma) f(\gamma) d\gamma = a^{tt} \left\{ \int_0^t U_{tt}(x, t-\gamma) f(\gamma) d\gamma + [U(x, t_1)] f'(t-t_1) + [U_t(x, t_1)] f(t-t_1) \right\}.$$

To perform the differentiations called for in the first-order terms in (3.4) we may use the results already obtained. For $k = 1, 2, \dots, n-1$,

$$(3.8) \quad b^k \frac{\partial}{\partial x_k} \int_0^t U(x, t-\gamma) f(\gamma) d\gamma = b^k \left\{ \int_0^t U_k(x, t-\gamma) f(\gamma) d\gamma - [U(x, t_1)] f(t-t_1) \psi_k \right\}.$$

For $k = n$ we have

$$(3.9) \quad b^t \frac{\partial}{\partial t} \int_0^t U(x, t-\gamma) f(\gamma) d\gamma = b^t \left\{ \int_0^t U_t(x, t-\gamma) f(\gamma) d\gamma + [U(x, t_1)] f(t-t_1) \right\}.$$

For the remaining term in (3.4) we merely restate that

$$(3.10) \quad c \int_0^t U(x, t-\gamma) f(\gamma) d\gamma = c \int_0^t U(x, t-\gamma) f(\gamma) d\gamma.$$

To verify the correctness of the Duhamel principle, that is, that (3.4) is correct, we must now add up the terms in equations (3.5) to (3.10). We find on the right side that the terms involving integrals amount to

$$\int_0^t L(U(x, t-\gamma)) f(\gamma) d\gamma.$$

Since $U(x, t)$ is required to satisfy $L(u) = 0$ for $t-\gamma < t_1$ and $t-\gamma > t_1$, these terms add up to zero.

We collect next those terms on the right side of equations (3.5) to (3.10) involving $f'(t-t_1)$. These are

$$f'(t-t_1) \left\{ \sum_{i,j=1}^{n-1} a^{ij} [U(x, t_1)] \psi_i \psi_j - \sum_{i=1}^{n-1} a^{it} [U(x, t_1)] \psi_i - \sum_{j=1}^{n-1} a^{tj} [U(x, t_1)] \psi_j + a^{tt} [U(x, t_1)] \right\}.$$

We recall that $t = \psi(x_1, \dots, x_{n-1})$ is an explicit form of $\phi(x_1, x_2, \dots, x_n) = 0$. Hence $\psi_i = -\frac{\phi_i}{\phi_t}$. We see therefore that these terms amount to

$$\frac{f'(t-t_1)}{\phi_t^2} [U(x, t_1)] \sum_{i,j=1}^n a^{ij} \phi_i \phi_j .$$

We require that

$$(3.11) \quad \sum_{i,j=1}^n a^{ij} \phi_i \phi_j = 0 \text{ on } \phi = 0;$$

that is, if $\phi = 0$ is a surface on which $U(x, t)$ is discontinuous, then ϕ is required to satisfy this first-order differential condition on $\phi = 0$. This condition is known as the characteristic condition⁵.

We collect next the terms involving $f(t-t_1)$. Omitting this factor these terms are:

$$\begin{aligned} & - \sum_{i,j=1}^{n-1} a^{ij} [U(x, t_1)] \psi_{ij} - \sum_{i,j=1}^{n-1} a^{ij} [U_i(x, t_1)] \psi_j - \sum_{i,j=1}^{n-1} a^{ij} [U_j(x, t_1)] \psi_i \\ & - \sum_{i,j=1}^{n-1} a^{ij} [U_t(x, t_1)] \psi_i \psi_j + \sum_{i=1}^{n-1} a^{it} [U_i(x, t_1)] + \sum_{j=1}^{n-1} a^{jt} [U_j(x, t_1)] \\ & + a^{tt} [U_t(x, t_1)] - \sum_{k=1}^{n-1} b^k [U(x, t_1)] \psi_k + b^t [U(x, t_1)] . \end{aligned}$$

We shall show later, in connection with the first discontinuity condition of section V, that these terms can be rewritten as on the left side of (3.12). We now require that

$$(3.12) \quad \sum_{i,j=1}^n [\bar{U}] a^{ij} \phi_{ij} + \sum_{k=1}^n [U] b^k \phi_k + 2 \sum_{i,j=1}^n a^{ij} [U_i] \phi_j = 0 \text{ on } \phi = 0.$$

Conditions (3.11) and (3.12) are imposed on $U(x, t)$ and $\overset{\text{on}}{\nearrow}$ the discontinuity surface $\phi = 0$ in order that the Duhamel principle hold⁶.

We must now verify that the Duhamel principle holds for points x on the boundary surface S_1 . On this surface $u(x, t) = h(x)f(t)$, whereas $U(x, t) = h(x)\eta(t)$. Then

$$h(x)f(t) = \frac{\partial}{\partial t} \int_0^t h(x) \eta(t-\tau) f(\tau) d\tau.$$

Since $\eta(t-\tau) = 1$ for $0 < \tau < t$, the verification is immediate not only for $t > 0$ but as $t \rightarrow 0+$. Likewise the initial conditions are satisfied.

5. See Courant, R. and Hilbert, D., loc. cit., V. II, p. 351. The significance of (3.11) and (3.12) will be clearer after the discussion of section V of this paper. See the discussion of equation (5.12) and see (5.15).

6. See the preceding footnote.

We consider next the modifications needed in the above verification of the Duhamel principle to apply it to Problem II. Equations (3.5) to (3.10) are still correct statements. Moreover we shall again impose conditions (3.11) and (3.12), so that those terms which were previously discarded on the right sides are again discarded. However, the sum of those terms on the left sides of equations (3.5) to (3.10) which involve an integral now yields, in view of the non-homogeneous term in (2.2) and step (3.3),

$$\int_0^t h(x)f(s)ds.$$

The terms on the right side of steps (3.5) to (3.10) which contain the integral of U and its derivatives amount to

$$(3.13) \quad \int_0^t L(U(x,t-\tau))f(\tau)d\tau.$$

Since U is required to satisfy the equation $L(U) = h(x)\eta(t)$, the quantity (3.13) equals

$$\int_0^t h(x)\eta(t-\tau)f(\tau)d\tau.$$

Since $\eta(t-\tau)$ is 1 for $0 < \tau < t$, we see that the Duhamel integral for $u(x,t)$ satisfies the differential equation.

As to the initial and boundary conditions, satisfaction of the differential equation takes care of points x not only outside of V but also in V , for $h(x) \neq 0$ only in V . Given, also, a boundary condition that $u(x,t) = 0$ on some surface S_1 , we require that $U(x,t)$ satisfy the same condition. It follows at once from (3.1) that for x on S_1 , the left and right sides are both zero for all $t > 0$ and as t approaches $0+$.

IV. Derivation of the Asymptotic Expansion

We now restrict the time behavior of the $f(t)$ which appears in the boundary condition of Problem I and the differential equation of Problem II to $e^{-i\omega t}$ and we use the Duhamel integral to derive the asymptotic expansion for $u(x,t)$. Strictly we shall show that $u(x,t)$ contains a "transient" term which approaches zero as t becomes infinite and that the remaining "steady state" part of $u(x,t)$ is of the form $\bar{u}(x_1, x_2, \dots, x_{n-1})e^{-i\omega t}$. We then derive an asymptotic series for $\bar{u}$.

We begin with the Duhamel integral, wherein we let $f(t)$ be $e^{-i\omega t}$. Then

$$(4.1) \quad u(x,t) = \frac{\partial}{\partial t} \int_0^t U(x, t-\tau) e^{-i\omega \tau} d\tau.$$

Let $t-\tau = s$ and $t = t$. With this change of variable

$$\begin{aligned} u &= \frac{\partial}{\partial t} \int_0^t U(x, s) e^{-i\omega(t-s)} ds, \\ &= \frac{\partial}{\partial t} (e^{-i\omega t} \int_0^t U(x, s) e^{i\omega s} ds), \\ &= U(x, t) - i\omega e^{-i\omega t} \int_0^t U(x, s) e^{i\omega s} ds. \end{aligned}$$

Adding and subtracting equivalent terms gives

$$\begin{aligned} u &= U(x, t) - U(x, \infty) - i\omega e^{-i\omega t} \int_0^t (U(x, s) - U(x, \infty)) e^{i\omega s} ds + U(x, \infty) e^{-i\omega t}, \\ &= U(x, t) - U(x, \infty) + e^{-i\omega t} \left\{ U(x, \infty) - i\omega \int_0^t (U(x, s) - U(x, \infty)) e^{i\omega s} ds \right\}. \end{aligned}$$

We now let t become infinite. We shall neglect what may be considered on physical grounds as a transient, namely the terms $U(x, t) - U(x, \infty)$, because at any point x , the field due to the pulse should establish itself in a "short" time and readily approach a static condition. For the same physical reason too we expect that the integrand should approach zero rapidly for large s and hence that the integral from 0 to ∞ should not only exist but differ little from the integral from 0 to t . Hence with increasing time, $u(x, t)$ takes the form $\bar{u}(x)e^{-i\omega t}$, and

$$\bar{u} = U(x, \infty) - i\omega \int_0^\infty (U(x, \tau) - U(x, \infty)) e^{i\omega \tau} d\tau.$$

We now integrate by parts to obtain the asymptotic expansion for $\bar{u}$. One such integration yields

$$\bar{u} = U(x, \infty) - \left(U(x, \gamma) - U(x, \infty) \right) e^{i\omega\gamma} \Big|_0^\infty + \int_0^\infty U_\gamma(x, \gamma) e^{i\omega\gamma} d\gamma.$$

In evaluating the second term we must take into account the discontinuities in $U(x, t)$ at values of t . We have assumed that these discontinuities are finite. There may be an infinite number of them. Let $\gamma_1, \gamma_2, \dots, \gamma_\alpha, \dots$ be these discontinuities and let $[U(x, \gamma_\alpha)] = U(x, \gamma_\alpha+) - U(x, \gamma_\alpha-)$. Then

$$\bar{u} = U(x, 0) + \sum_\alpha [U(x, \gamma_\alpha)] e^{i\omega\gamma_\alpha} + \int_0^\infty U_\gamma(x, \gamma) e^{i\omega\gamma} d\gamma.$$

Another integration by parts yields

$$\bar{u} = U(x, 0) + \sum_\alpha [U(x, \gamma_\alpha)] e^{i\omega\gamma_\alpha} - \frac{1}{i\omega} \sum_\alpha [U_\gamma(x, \gamma_\alpha)] e^{i\omega\gamma_\alpha} + \frac{1}{i\omega} \int_0^\infty U_{\gamma\gamma}(x, \gamma) e^{i\omega\gamma} d\gamma.$$

The second summation includes $U_\gamma(x, 0)$ and $U_\gamma(x, \infty)$. We may expect on physical grounds that $U(x, \infty)$ is zero, because $U(x, t)$ approaches a static field as t increases. The quantity $U(x, 0)$ should also be zero at points x away from the source, since $U(x, t)$ is 0 there for t less than some positive constant. Hence we discard these values of t_α in the second summation.

It is apparent that the process of integration by points can be continued and that we obtain a series in positive integral powers of $\frac{1}{\omega}$ for $\bar{u}$. We therefore write

$$(4.2) \quad \bar{u}(x) = U(x, 0) + \sum_\alpha [U(x, \gamma_\alpha)] e^{i\omega\gamma_\alpha} - \frac{1}{i\omega} \sum_\alpha [U_\gamma(x, \gamma_\alpha)] e^{i\omega\gamma_\alpha} + \frac{1}{(i\omega)^2} \sum_\alpha [U_{\gamma\gamma}(x, \gamma_\alpha)] e^{i\omega\gamma_\alpha} \dots$$

For points x not on S_1 or at the source, $U(x, 0)$ equals 0 and may be neglected.

This infinite series is certainly asymptotic to $\bar{u}$ since the integral neglected after any stage is seen by an integration by parts to be of degree one higher in $\frac{1}{\omega}$ than the preceding ones. The argument assumes that the integral converges. This may be expected on physical grounds⁷, because the higher derivatives of $U(x, t)$ should certainly approach zero rapidly with increasing t . If we stop the integration by parts at any one term we obtain an exact expression for $\bar{u}(x)$ wherein the remainder, that is, the integral, is of order one higher in $\frac{1}{\omega}$ than the last term of the finite series.

At any x , the γ_α in (4.2) are the values of t at which $U(x, t)$ or any one of its time-derivatives is discontinuous. We have already noted in connection with the verification of the Duhamel integral that if (x, t) is a point in the n -dimensional space

7. The argument here should later be replaced by a mathematical one which would enable us to decide from a knowledge of the differential equation itself and the initial and boundary conditions whether the discontinuities in the time derivatives of $U(x, t)$ remain finite and that these time derivatives approach zero rapidly.

at which $U(x,t)$ is discontinuous, then (x,t) lies on a hypersurface $\phi = 0$ which we have required to satisfy the characteristic condition (3.11). It will be shown later⁸ that the points (x,t) at which U_t or any higher time-derivative is discontinuous likewise lie on $\phi = 0$. Hence given an x , the corresponding γ_α may be determined by solving $\phi = 0$ for t . There may of course be many γ_α for a given x and we shall therefore denote the explicit solutions by $\gamma_\alpha = \psi_\alpha(x_1, x_2, \dots, x_{n-1})$. Hence we may replace the γ_α in (4.2) by the corresponding ψ_α .

It is obvious from the form of (4.2) that the asymptotic series should be useful for large ω and that for $\omega = \infty$, the first term yields the geometrical optics field in the case of electromagnetic waves and the corresponding limits in the cases of other physical phenomena. Actually the series (4.2) gives more than the usual geometrical optics methods do, for the phase factors $e^{i\omega\gamma_\alpha}$ are also included.

To use the series (4.2) one must know the coefficients. We have already indicated that we know the γ_α . However the essential part of each coefficient is a discontinuity or sum of discontinuities of $U(x,t)$ or one of its time-derivatives. These are generally not known at various points (x,t) ; however it may be possible to find them without requiring a knowledge of the entire solution $U(x,t)$. We shall therefore undertake a general study of discontinuities of solutions of equations (2.1) and (2.2), from which we shall obtain ordinary differential equations for each of these discontinuities. These differential equations may then be used to determine the discontinuities directly and thereby the asymptotic expansion itself.

8. See section VI.

V. Derivation of the First Discontinuity Condition and First Transport Equation

We proceed to study discontinuities of solutions of the differential equations (2.1) and (2.2). Of course at a point or along a hypersurface on which u or its first or second derivatives are discontinuous the differential equations have no meaning. Moreover, if some discontinuity surface separates a region G_1 in which (2.1) holds from a region G_2 in which (2.1) holds, there is no essential connection between the two solutions in the two regions. Initial and boundary conditions may determine the solution of (2.1) uniquely in region G_1 but the solution in G_2 may still be independent of the one in G_1 . Evidently to discuss discontinuous solutions of these differential equations, some relationship must be imposed to connect the solutions on either side of a discontinuity surface. We shall discuss equation (2.1) and remark later on the minor additional features required to treat (2.2).

One can require that a discontinuous solution of (2.1) be a limit of continuous solutions each of which satisfies (2.1) in both regions and on the discontinuity surface⁹. The discontinuous solution may then be expressed in terms of the Heaviside unit function and substituted in the differential equation. Under this procedure derivatives of the unit function are the δ -function and its derivatives. By suitable operation with this function it is possible to obtain the discontinuity condition obtained below and even to secure the higher discontinuity conditions obtained in the next section¹⁰. The use of the δ -function and its derivatives can be made mathematically rigorous by using the theory of distributions due principally to Laurent Schwartz¹¹.

To achieve rigor and to avoid dependence upon the somewhat elaborate theory of distributions we shall employ another procedure. We shall replace the differential equation (2.1) by an integral equation which is equivalent to (2.1) in a region G in which the first and second derivatives of u exist. The integral will be meaningful, however, even for discontinuous u , and we shall require that the discontinuous solutions u considered in this paper satisfy the integral equation. From this equation we shall derive discontinuity conditions which must hold on any discontinuity surface. These discontinuity conditions will then be converted into ordinary differential equations or transport equations for the discontinuities in u , u_t , u_{tt} , etc. The solutions of these differential equations corresponding to suitable initial conditions are the coefficients of the asymptotic series (4.2). In this section we shall obtain the first discontinuity condition and first transport equation.

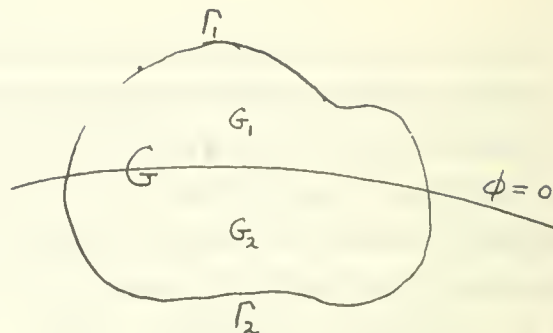
9. See Courant-Hilbert, loc. cit., p. 360.

10. For the case of the wave equation, the first discontinuity condition is obtained by the procedure just described in Bremmer, H., loc. cit. See the last part of this section.

11. Schwartz, L., Théorie des distributions, 2 vols., Hermann and Co., Paris, 1951-52.

Let u and its first and second derivatives exist and be bounded in a region G . Let Ω be any arbitrary function of $x_1, x_2, \dots, x_n, x_n=t$, such that Ω and all first-order derivatives vanish on the boundary and such that Ω possesses derivatives of any finite order with respect to the x_i in G . Let $dv=dx_1 dx_2 \dots dx_n$. Then

$$(5.1) \quad \int_G \Omega L(u) dv = \int_G \Omega \sum_{i,j} a^{ij} u_{ij} dv + \int_G \Omega \sum_k b^k u_k dv + \int_G \Omega c u dv.$$



Let (dx_j) indicate that partial integration with respect to dx_j has been performed. Then by integration by parts

$$\int_G \Omega \sum_{i,j} a^{ij} u_{ij} dv = \int_{\Gamma} \sum_{i,j} \Omega a^{ij} u_i (dx_j) - \int_G \sum_{i,j} (\Omega a^{ij})_j u_i dv.$$

Since Ω vanishes on Γ , the surface integral vanishes. Another integration by parts plus the fact that Ω and Ω_j vanish on Γ yields

$$(5.2) \quad \int_G \Omega \sum_{i,j} a^{ij} u_{ij} dv = \int_G \sum_{i,j} (\Omega a^{ij})_{ij} u dv.$$

Likewise

$$(5.3) \quad \int_G \Omega \sum_k b^k u_k dv = - \int_G \sum_k (\Omega b^k)_k u dv.$$

We may therefore state from (5.2) and (5.3) that

$$(5.4) \quad \int_G \Omega L(u) dv = \int_G \sum_{i,j} u (\Omega a^{ij})_{ij} dv - \int_G \sum_k u (\Omega b^k)_k dv + \int_G \Omega c u dv.$$

Now the adjoint differential equation of $L(u)$ is, by definition,¹²

$$M(\Omega) = \sum_{i,j} (\Omega a^{ij})_{ij} - \sum_k (\Omega b^k)_k + c \Omega.$$

Hence (5.4) tells us that

$$(5.5) \quad \int_G \Omega L(u) dv = \int_G u M(\Omega) dv.$$

We conclude also from the above steps that in a region G wherein u has all first and second derivatives which are bounded, the differential equation $L(u)=0$ may be converted into the integral equation¹³

$$(5.6) \quad \int_G u M(\Omega) dv = 0,$$

12. Courant-Hilbert, loc.cit., p.434.

13. cf. Courant-Hilbert, loc.cit., pp. 469-70.

wherein it is understood that u must satisfy the integral equation for any Ω which possesses first and second derivatives in G and is such that Ω and its first derivatives vanish on the boundary of G . Moreover, since the steps leading from (5.1) to (5.5) are reversible by virtue of the arbitrariness of Ω , the differential equation $L(u) = 0$ is entirely equivalent to the integral equation (5.6).

If G is a region in which u or any of its first or second derivatives are discontinuous, then the equation $L(u) = 0$ has no meaning throughout G . However, equation (5.6) does, and we therefore agree to replace the differential equation by the more general integral equation (5.6). We note that the use of (5.6) as our starting point implies that $L(u) = 0$ holds in any subregion of G in which u and its first and second derivatives exist, for we may take a hypersphere about any point in this subregion, choose an Ω which with its first derivatives vanishes on this hypersphere, and apply the steps leading from (5.6) to (5.1). We require not only that a discontinuous u satisfy the integral equation (5.6) but that u and its first and second derivatives be continuous from within and bounded over those closed subregions in which u and its derivatives exist. Otherwise stated, u and its first and second derivatives must be continuous except on a surface $\phi(x_1, \dots, x_n) = 0$ on which all discontinuities of u and its successive time-derivatives may have only jump discontinuities. It is assumed also that the first and second derivatives of ϕ exist.

The conditions on u and on Ω specified above suffice to derive the discontinuity conditions and first transport equation in this section. However, to derive later the discontinuity conditions and transport equations for $[u_t]$, $[u_{tt}]$, etc., we require the further conditions that within any bounded region G , there exist a finite number of closed subdomains within which u_t , u_{tt} , etc. and their first and second derivatives with respect to all variables be continuous and bounded. Moreover we require that Ω , Ω_t , Ω_{tt} , etc., and all their first derivatives vanish on the boundary of G , while these functions possess derivatives of all orders within G . These conditions on u are sufficiently broad to apply the results obtained in the remaining sections to the functions u considered in connection with the derivation of the asymptotic expansion.

We shall now derive discontinuity conditions which u must satisfy on any surface on which it is discontinuous. Let G therefore be a region in which u is discontinuous on some surface $\phi(x_1, \dots, x_n) = 0$ which divides G into two domains G_1 and G_2 . We shall assume that ϕ can be expressed in the form $t - \psi(x_1, \dots, x_{n-1})$. Let Γ_1 be that part of Γ which bounds G_1 , and let Γ_2 be that part which bounds G_2 .

We may regard u as consisting of two functions u_1 and u_2 , such that u_1 satisfies the integral equation in G_1 and is 0 in G_2 , and conversely for u_2 . Within G_1 , u_1 therefore satisfies $L(u) = 0$, and likewise for u_2 . In $G_1 + G_2$, $u = u_1 + u_2$ satisfies (5.6).

We seek conditions on $[u] = u_1 - u_2$ which hold on $\phi = 0$. We shall use the fact, even for discontinuous u , that $\int_G = \int_{G_1} + \int_{G_2}$.

Consider first a typical term of (5.6) for i and $j = 1, 2, \dots, n-1$. By integration by parts

$$\int_{G_1} u(a^{ij}\Omega)_{ij} dv = \int_{\Gamma_1+\phi} u(a^{ij}\Omega)_i (dx_j) - \int_{G_1} (a^{ij}\Omega)_i u_j dv.$$

Because Ω and Ω_i are 0 on Γ_1 we may ignore that part of the "surface" integral. In the integrand of the remaining surface integral we must understand that x_j has been replaced by the expression obtained in solving $\phi = 0$ for x_j .

By another integration by parts applied to the volume integral we obtain

$$\int_{G_1} u(a^{ij}\Omega)_{ij} dv = \int_{\phi} u(a^{ij}\Omega)_i (dx_j) - \int_{\phi} a^{ij}\Omega u_j (dx_i) + \int_{G_1} a^{ij}\Omega u_{ij} dv.$$

Let us note now that the direction cosines of the normal to $\phi = 0$ are

$$\frac{\phi_1}{\pm \sqrt{\sum_i \phi_i^2}} = \pm \lambda \phi_1.$$

We shall suppose that the plus sign in front of the radical indicates the direction of the normal points outside of G_1 and into G_2 . Also let α_i , $i = 1, 2, \dots, n$, denote the direction angles of this "positive" direction. Let us also use the notation dw for an element of surface on $\phi = 0$. We may rewrite the preceding result thus:

$$\int_{G_1} u(a^{ij}\Omega)_{ij} dv = \int_{\phi} u(a^{ij}\Omega)_i \lambda \phi_j dw - \int_{\phi} a^{ij}\Omega u_j \lambda \phi_i dw + \int_{G_1} a^{ij}\Omega u_{ij} dv.$$

Now $dw = dx_1 \dots dx_{n-1} \sec \alpha_n = dx_1 \dots dx_{n-1} \frac{1}{\lambda \phi_n}$. Since $w_j = -\phi_j/\phi_n$ we may rewrite the preceding result:

$$\int_{G_1} u(a^{ij}\Omega)_{ij} dv = -\int_{\phi} u(a^{ij}\Omega)_i \psi_j (dx_n) - \int_{\phi} a^{ij}\Omega u_j \lambda \phi_i dw + \int_{G_1} a^{ij}\Omega u_{ij} dv.$$

In the first surface integral we must remember that x_n is replaced by its value $w(x_1, \dots, x_{n-1})$. However the derivative with respect to x_i applies only to the x_i present before the substitution.

To achieve integrands in which only Ω and Ω_t appear - the purpose will be apparent later - we make a further transformation of this surface integral. We note that

$$\frac{\partial}{\partial x_1} (a^{ij}\Omega) = (a^{ij}\Omega)_i + (a^{ij}\Omega)_t \psi_1.$$

We substitute for $(a^{ij}\Omega)_i$ in the first surface integral on the right and integrate by parts with respect to x_i the integrand containing $\frac{\partial}{\partial x_i} (a^{ij}\Omega)$. The first term which results from the integration by parts must be taken over the boundary of ϕ , which boundary lies on the boundary Γ of G . Since the integrand contains Ω , which vanishes on Γ , this integral vanishes. There results

$$\int_{G_1} u(a^{ij}\Omega)_{ij} dv = \int_{\phi} a^{ij}\Omega \frac{\partial}{\partial x_i} (u\psi_j)(dx_n) + \int_{\phi} (a^{ij}\Omega)_t u\psi_i\psi_j(dx_n) - \int_{\phi} a^{ij}\Omega u_j\lambda\phi_i dw + \int_{G_1} a^{ij}\Omega u_{ij} dv.$$

We now make simple changes to obtain

$$(5.7) \quad \int_{G_1} u(a^{ij}\Omega)_{ij} dv = \int_{\phi} a^{ij}\Omega \left\{ (u_i + u_t\psi_i)\psi_j + u\psi_{ij} \right\} \lambda\phi_t dw + \int_{\phi} (a^{ij}\Omega)_t u\psi_i\psi_j\lambda\phi_t dw - \int_{\phi} a^{ij}\Omega u_j\lambda\phi_i dw + \int_{G_1} a^{ij}\Omega u_{ij} dv.$$

We now have the useful form of those terms of (5.6) in which i and j run from 1 to $n-1$. We consider next those terms in which the derivatives with respect to x_i and $x_n (=t)$ appear:

$$(5.8) \quad \int_{G_1} (\Omega a^{in})_{in} u dv = \int_{\phi} (a^{in}\Omega)_t u\lambda\phi_i dw - \int_{G_1} (a^{in}\Omega)_t u_i dv;$$

$$\int_{G_1} (\Omega a^{in})_{in} u dv = \int_{\phi} (a^{in}\Omega)_t u\lambda\phi_i dw - \int_{\phi} a^{in}\Omega u_i\lambda\phi_t dw + \int_{G_1} a^{in}\Omega u_{in} dv.$$

We note that either order of performing the integration by parts with respect to x_i and x_n gives the same result.

We consider next that term in (5.6) which involves $\frac{\partial^2}{\partial x_n^2}$:

$$(5.9) \quad \int_{G_1} (a^{nn}\Omega)_{nn} u dv = \int_{\phi} (a^{nn}\Omega)_n u\lambda\phi_w - \int_{G_1} (a^{nn}\Omega)_n u_n dv;$$

$$\int_{G_1} (a^{nn}\Omega)_{nn} u dv = \int_{\phi} (a^{nn}\Omega)_n u\lambda\phi_t dw - \int_{\phi} a^{nn}\Omega u_n\lambda\phi_t dw + \int_{G_1} a^{nn}\Omega u_{nn} dv.$$

The first-order terms in (5.6) are treated as follows:

$$(5.10) \quad - \int_{G_1} (\Omega b^k)_k u dv = - \int_{\phi} \Omega b^k u\lambda\phi_k dw + \int_{G_1} \Omega b^k u_k dv.$$

Finally, the $c\Omega u$ term in (5) is left as is. That is,

$$(5.11) \quad \int_{G_1} c \Omega u dv = \int_{G_1} c \Omega u dv.$$

All of the steps from (5.7) to (5.11) could be performed with respect to G_2 instead of G_1 . The only difference is that λ would be replaced by $-\lambda$ because the normal points out of G_2 into G_1 . We now apply (5.6) remembering that $\int_G = \int_{G_1} + \int_{G_2}$. We note before collecting the results in equations (5.7) to (5.11) that the volume-integral terms remaining on the right side add up to $+\int_{G_1} \Omega L(u)$. Since $L(u)$ holds within G_1 and all quantities in $L(u)$ approach finite values on the boundary of G_1 , this volume integral adds up to zero. Similarly for $\int_{G_2} \Omega L(u)$. Further, since the results over G_1 and G_2 are the same except for sign we may combine them. However those values of u and its derivatives which obtain on $\phi = 0$ when approached from G_1 may differ from those which obtain when approached from G_2 . We shall use the notation $[u]$, $[u_1]$, etc., to indicate the differences $u_1 - u_2$, $u_{11} - u_{21}$, etc.

The result of combining the terms in (5.7) to (5.11) with the corresponding terms for G_2 will of course be 0 because of (5.6). Also since we are concerned in the paper as a whole with the case in which the coefficients of $L(u)$ are independent of $x_n = t$, we shall use this fact. We note that the terms which appear on the right sides of equations (5.7) to (5.11) contain either Ω or Ω_t as a factor. We argue that Ω and Ω_t may be made 0 on $\phi = 0$ independently of each other while retaining the otherwise arbitrary nature of the Ω -function. In view of this fact the coefficient of Ω and the coefficient of Ω_t must each be zero in order that the surface integral be 0. We consider first the terms in Ω_t . These all contain $|u|$ as a factor. If $|u|$ is not zero then we find that on $\phi = 0$

$$\sum_{i,j=1}^{n-1} a^{ij} \psi_i \psi_j \phi_t + 2 \sum_{i=1}^{n-1} a^{in} \phi_i + a^{nn} \phi_t = 0.$$

We recall that $\phi = t - \psi$. Hence $\phi_t = 1$, $\phi_i = -\psi_i$. We see therefore that these terms amount to

$$(5.12) \quad \sum_{i,j=1}^n a^{ij} \phi_i \phi_j = 0 \quad \text{on } \phi = 0.$$

In other words, the discontinuity surface $\phi = 0$ must satisfy the condition (5.12) if u is discontinuous there. Equation (5.12) is known as the characteristic condition¹⁴. Equation (5.12) is not strictly a partial differential equation since it need hold only on $\phi = 0$. However we may convert it to a proper differential equation in ψ by using $\phi = t - \psi = 0$.

¹⁴. For a full discussion of the characteristic condition see Courant-Hilbert, loc. cit., V.II, Chap. 6, article 1.

We now consider those terms in equations (5.7) to (5.11) which contain Δ . Since these terms must also add up to zero we may write

$$(5.13) \quad \sum_{i,j=1}^{n-1} [u] a^{ij} \psi_{ij} \phi_t + \sum_{i,j=1}^{n-1} a^{ij} [u_i] \psi_j \phi_t + \sum_{i,j=1}^{n-1} [u_t] \psi_i \psi_j \phi_t - \sum_{i,j=1}^{n-1} a^{ij} [u_j] \phi_i \\ - 2 \sum_{i=1}^{n-1} a^{in} [u_i] \phi_t - a^{nn} [u_n] \phi_t - \sum_{k=1}^n [u] b^k \phi_k = 0.$$

We recall the fact that $\phi = t - \psi$, and hence that $\phi_t = 1$, $\phi_{tt} = 0$, $\phi_i = -\psi_i$, $\phi_{ij} = -\psi_{ij}$, $\phi_{it} = 0$. We use these facts to rewrite (5.13) as

$$(5.14) \quad - \sum_{i,j=1}^n [u] a^{ij} \phi_{ij} - \sum_{i,j=1}^{n-1} a^{ij} [u_i] \phi_j + \sum_{i,j=1}^{n-1} a^{ij} [u_t] \phi_i \phi_j - \sum_{i,j=1}^{n-1} a^{ij} [u_j] \phi_i \\ - 2 \sum_{i=1}^{n-1} a^{in} [u_i] \phi_t - a^{nn} [u_n] \phi_t - \sum_{k=1}^n [u] b^k \phi_k = 0.$$

The second and fourth terms combine. We may use equation (5.12) to replace the third term by $[u_t] (-2 \sum_{i=1}^{n-1} a^{in} \phi_i \phi_t - a^{nn} \phi_t \phi_t)$. The result thus far is

$$(5.15) \quad - \sum_{i,j=1}^n [u] a^{ij} \phi_{ij} - 2 \sum_{i,j=1}^n a^{ij} [u_i] \phi_j - \sum_{k=1}^n [u] b^k \phi_k = 0 \quad \text{on } \phi = 0.$$

This result may be called the first discontinuity condition (aside from the characteristic condition). It may be converted into an ordinary differential equation in $[u]$ as follows. We introduce the concept of differentiation in the direction transverse to $\phi = 0$. The direction itself¹⁵ is given by

$$(5.16) \quad \dot{x}_i = \frac{dx_i}{ds} = \sum_{k=1}^n a^{ik} \phi_k = \frac{\partial \Phi}{\partial x_i},$$

where $\Phi = \sum_{i,j=1}^n a^{ij} \phi_i \phi_j$.

The derivative of a function $u(x_1, \dots, x_n)$ in this transverse direction is then given by the directional derivative

$$\frac{\partial u}{\partial s} = \sum_{i=1}^n \frac{\partial u}{\partial x_i} \dot{x}_i = \sum_{i,j=1}^n a^{ij} \frac{\partial u}{\partial x_i} \phi_j = \sum_{i,j=1}^n a^{ij} u_i \phi_j.$$

We may therefore rewrite the first discontinuity condition (5.15), after a change in sign, as

15. See Courant-Hilbert, loc.cit., p.353. Since the normal to $\phi=0$ has the direction numbers ϕ_i , $i=1,2,\dots,n$, the transverse direction is perpendicular to the normal and therefore lies in the surface $\phi = 0$.

$$(5.17) \quad [u] \left(\sum_{i,j=1}^n a^{ij} \phi_{ij} + \sum_{k=1}^n b^k \phi_k \right) + 2 \frac{\partial [u]}{\partial s} = 0 \quad \text{on } \phi = 0.$$

This ordinary differential equation will be called the (first) transport equation¹⁶. As s traverses a path which has at each point the direction of the transverse directional derivative (5.16), equation (5.17) gives the behavior of the discontinuity $[u]$ along this path. It will be noted that (5.17) is an ordinary linear homogeneous differential equation. The equations (5.16) are called the equations of the characteristic rays and they are the rays of ordinary optics.

We may abbreviate the writing of (5.17) by introduction of the operator $\bar{L}(u) = \sum_{i,j=1}^n a^{ij} u_{ij} + \sum_{k=1}^n b^k u_k$. Then (5.17) reads as follows¹⁷:

$$(5.18) \quad \bar{L}(\phi) [u] + 2 \frac{\partial [u]}{\partial s} = 0.$$

We shall specialize the characteristic equation, the discontinuity condition, and the transport equation to the three-dimensional wave equation. For the wave equation

$$u_1^2 + u_2^2 + u_3^2 - n^2 u_{tt}^2 = 0$$

and for $\phi = t - \psi$, the characteristic equation (5.12) becomes the true partial differential equation

$$(5.19) \quad \sum_{i=1}^3 \psi_i^2 = n^2.$$

The first discontinuity condition (5.15) becomes

$$(5.20) \quad -[u] \left(\sum_{i=1}^3 \psi_{ii} \right) + 2 \left(-\sum_{i=1}^3 [u_i] \psi_i - n^2 [u_t] \right) = 0,$$

wherein t is replaced by ψ . If we use vector notation and remember that in (5.20) t is to be replaced by ψ , then we may rewrite (5.20) as

$$(5.21) \quad \Delta \psi [u] + 2 \text{grad}[u] \cdot \text{grad } \psi = 0. \quad 18$$

The equation (5.18) for the rays may also be specialized to yield

16. Had we not assumed that the coefficients of $L(u) = 0$ were independent of x_n we would have obtained from the derivation in this section the slightly more general transport equation

$$[u] \left(\sum_{i,j=1}^n a^{ij} \phi_{ij} + \sum_{k=1}^n b^k \phi_k - \sum_{i,j=1}^n a_t^{ij} \phi_i \phi_j \right) + 2 \frac{\partial [u]}{\partial s} = 0.$$

Later theory in this paper does not carry over to this more general case.

17. See Courant-Hilbert, loc.cit., p.361.

18. This result was obtained by Bremmer, loc.cit., p.421 and also Lowell, loc.cit., p.282.

$$(5.22) \quad \begin{aligned} \dot{x}_i &= \frac{dx_i}{ds} = -\psi_i & , & & i = 1, 2, 3, \\ \frac{dt}{ds} &= -n^2 & . & \end{aligned}$$

One sees from (5.22) that it is both possible and convenient to replace the parameter s along the rays by the time t . Also, since the characteristic surface is given by $t = \tau$, we see from (5.22) that the rays are orthogonal to the family of characteristic surfaces in 3-space obtained by regarding t as a parameter.

All of the derivations of this section apply as well to the non-homogeneous equation (2.2) as to (2.1). If one considers a point (x, t) in space where no sources are present, that is, where the non-homogeneous term is zero or even continuous, no additional terms appear in the characteristic equation or discontinuity condition¹⁹.

19. To take into account the discontinuity which the source term $h(x)f(t)$ introduces on the discontinuity surface $\phi \equiv t = 0$, some slightly more involved considerations are necessary. See Kline, M., loc.cit., on the treatment of the source term.

VI. Derivation of the Higher Discontinuity Conditions and Higher Transport Equations

The starting point of our derivation of the first discontinuity condition and first transport equation was

$$(5.6) \quad \int_G uM(\Omega)dv = 0.$$

In this equation Ω is arbitrary except for the fact that it and its first derivatives vanish on the boundary Γ of G and Ω must possess derivatives of all finite orders in G . In view of the arbitrariness of Ω we may replace it by Ω_t , and we now utilize the additional requirement already mentioned in Section V that the first derivatives of Ω_t vanish on Γ . We find therefore that

$$(6.1) \quad \int_G uM(\Omega_t)dv = 0.$$

We shall now treat this integral equation as we treated (5.6). Consider first those terms which contain derivatives with respect to x_i and x_j , $i, j = 1, 2, \dots, n-1$, in addition to a derivative with respect to t . We shall treat a typical term

$$(6.2) \quad \int_{G_1} u(a^{ij}\Omega)_{tij}dv = \int_{\emptyset} u(a^{ij}\Omega)_{ij}(dx_n) - \int_{G_1} u_t(\Omega a^{ij})_{ij}dv.$$

In the surface integral it is understood that x_n or t is replaced by $\psi(x_1, \dots, x_{n-1})$, but the subscripts i, j imply differentiation which would obtain before the substitution. Now

$$(a^{ij}\Omega)_{ij} = \frac{\partial}{\partial x_j} (a^{ij}\Omega)_i - (a^{ij}\Omega)_{it} \psi_j.$$

We use this relation in (6.2) and obtain

$$(6.3) \quad \int_{G_1} u(a^{ij}\Omega)_{tij}dv = \int_{\emptyset} u \frac{\partial}{\partial x_j} (a^{ij}\Omega)_i (dx_n) - \int_{\emptyset} u(a^{ij}\Omega)_{it} \psi_j (dx_n) - \int_{G_1} u_t (\Omega a^{ij})_{ij} dv.$$

We apply integration by parts to the first surface integral on the right side, the integration being with respect to x_j . The first term which results, namely,

$$\int_{\text{Bdy of } \emptyset} u(a^{ij}\Omega)_i (dx_j)(dx_n),$$

vanishes because the boundary of $\emptyset$ is on Γ and Ω and Ω_i vanish on Γ . Hence we obtain from (6.3) the fact that

$$(6.4) \int_{G_1} u(a^{ij}\Omega)_{tij} dv = - \int_{\phi} (a^{ij}\Omega)_i \frac{\partial u}{\partial x_1}(dx_n) - \int_{\phi} u(a^{ij}\Omega)_{it}\psi_j(dx_n) - \int_{G_1} u_t(\Omega a^{ij})_{ij} dv.$$

We shall use next the relation

$$(a_{ij}\Omega)_i = \frac{\partial}{\partial x_1} (a^{ij}\Omega) - (a^{ij}\Omega)_{t\psi_1}.$$

Hence from (6.4),

$$\begin{aligned} \int_{G_1} u(a^{ij}\Omega)_{tij} dv &= - \int_{\phi} \frac{\partial}{\partial x_1} (a^{ij}\Omega) \frac{\partial u}{\partial x_j}(dx_n) + \int_{\phi} (a^{ij}\Omega)_{t\psi_1} \frac{\partial u}{\partial x_1}(dx_n) - \int_{\phi} u(a^{ij}\Omega)_{it}\psi_j(dx_n) \\ &\quad - \int_{G_1} u_t(\Omega a^{ij})_{ij} dv. \end{aligned}$$

We may integrate by parts with respect to x_1 in the first integral on the right side and use the fact that Ω is 0 on the boundary of ϕ . Then

$$\begin{aligned} (6.5) \int_{G_1} u(a^{ij}\Omega)_{tij} dv &= \int_{\phi} (a^{ij}\Omega) \frac{\partial^2 u}{\partial x_1 \partial x_j}(dx_n) + \int_{\phi} (a^{ij}\Omega)_{t\psi_1} \frac{\partial u}{\partial x_j}(dx_n) - \int_{\phi} u(a^{ij}\Omega)_{it}\psi_j(dx_n) \\ &\quad - \int_{G_1} u_t(\Omega a^{ij})_{ij} dv. \end{aligned}$$

To obtain terms in the surface integrals which involve only Ω and time-derivatives of Ω we use the relation

$$(a^{ij}\Omega_t)_i = \frac{\partial}{\partial x_1} (a^{ij}\Omega_t) - a^{ij}\Omega_{tt}\psi_1$$

in the third surface integral on the right side of (6.5). We integrate by parts the resulting surface integral which contains the complete partial derivative with respect to x_1 , use the fact that $\Omega_t = 0$ on the boundary of ϕ , and obtain for $i, j = 1, 2, \dots, n-1$,

$$\begin{aligned} (6.6) \int_{G_1} u(a^{ij}\Omega)_{tij} dv &= \int_{\phi} (a^{ij}\Omega) \frac{\partial^2 u}{\partial x_1 \partial x_j}(dx_n) + \int_{\phi} (a^{ij}\Omega)_{t\psi_1} \frac{\partial u}{\partial x_1}(dx_n) + \int_{\phi} (a^{ij}\Omega)_t \frac{\partial(u\psi_j)}{\partial x_1}(dx_n) \\ &\quad + \int_{\phi} u(a^{ij}\Omega)_{tt}\psi_1\psi_j(dx_n) - \int_{G_1} u_t(\Omega a^{ij})_{ij} dv. \end{aligned}$$

We now consider those terms in (6.1) which involve derivatives with respect to x_i and x_n , $i=1, 2, \dots, n-1$, and a second derivative with respect to x_n or t . We treat a typical term, namely

$$\int_{G_1} u(a^{it}\Omega)_{itt} = \int_{\emptyset} u(a^{it}\Omega)_{it} (dx_n) - \int_{G_1} (a^{it}\Omega)_{it} u_t dv.$$

Since

$$(a^{it}\Omega)_{it} = \frac{\partial}{\partial x_i} (a^{it}\Omega_t) - a^{it}\Omega_{tt} \psi_i,$$

$$\int_{G_1} u(a^{it}\Omega)_{itt} = \int_{\emptyset} u \frac{\partial}{\partial x_i} (a^{it}\Omega_t) (dx_n) - \int_{\emptyset} u a^{it}\Omega_{tt} \psi_i (dx_n) - \int_{G_1} (a^{it}\Omega)_{it} u_t dv.$$

We integrate the first surface integral by parts and use the fact that Ω_t is 0 on the boundary of $\emptyset$. Then

$$(6.7) \int_{G_1} u(a^{it}\Omega)_{itt} = - \int_{\emptyset} a^{it}\Omega \frac{\partial u}{\partial x_i} (dx_n) - \int_{\emptyset} u a^{it}\Omega_{tt} \psi_i (dx_n) - \int_{G_1} (a^{it}\Omega)_{it} u_t dv.$$

We consider next the term in (6.1) which involves three differentiations with respect to x_n :

$$(6.8) \int_{G_1} u(a^{tt}\Omega)_{ttt} = \int_{\emptyset} u a^{tt}\Omega_{tt} (dx_n) - \int_{G_1} u_t a^{tt}\Omega_{tt} dv.$$

We consider next the second-order terms in (6.1), treating first those involving differentiation with respect to x_k , $k=1,2,\dots,n-1$, and x_n :

$$- \int_{G_1} u(b^k\Omega)_{kt} dv = - \int_{\emptyset} (b^k\Omega)_k u (dx_n) + \int_{G_1} (b^k\Omega)_k u_t dv.$$

Using $(b^k\Omega)_k = \frac{\partial}{\partial x_k} (b^k\Omega) - b^k\Omega_t \psi_k$, we obtain

$$- \int_{G_1} u(b^k\Omega)_{kt} dv = - \int_{\emptyset} \frac{\partial}{\partial x_k} (b^k\Omega) u (dx_n) + \int_{\emptyset} b^k\Omega_t \psi_k u (dx_n) + \int_{G_1} (b^k\Omega)_k u_t dv.$$

Integration by parts in the first surface integral plus the fact that Ω is 0 on the boundary of $\emptyset$ yields

$$(6.9) - \int_{G_1} u(b^k\Omega)_{kt} dv = \int_{\emptyset} b^k\Omega \frac{\partial u}{\partial x_k} (dx_n) + \int_{\emptyset} b^k\Omega_t \psi_k u (dx_n) + \int_{G_1} (b^k\Omega)_k u_t dv.$$

As for the second-order term in (6.1) involving two differentiations with respect to x_n ,

$$(6.10) - \int_{G_1} u(b^t\Omega)_{tt} dv = - \int_{\emptyset} (b^t\Omega)_t u (dx_n) + \int_{G_1} (b^t\Omega)_t u_t dv.$$

The first-order term in (6.1) is treated as follows:

$$(6.11) \int_{G_1} \Omega_t c u d\nu = \int_{\emptyset} \Omega c u (dx_n) - \int_{G_1} \Omega c u_t d\nu.$$

We intend to add the results of steps (6.6) to (6.11) to obtain an expression for $\int_{G_1} u M(\Omega_t) d\nu = 0$. We then intend to do likewise for $\int_{G_2} u M(\Omega_t) d\nu = 0$ and add this result to the first one. This sum will contain on the right side $-\int_G u_t M(\Omega) d\nu$. Now this volume integral is precisely (5.6) except that u_t here replaces u there. We shall therefore use the results of section V (equations (5.12) and (5.18) in particular) to write that

$$(6.12) -\int_G u_t M(\Omega) d\nu = -\int_{\emptyset} [u_t] \Omega_t \sum_{i,j=1}^n a^{ij} \phi_i \phi_j \lambda dw + \int_{\emptyset} \Omega (\bar{L}(\phi) [u_t] + 2 \frac{\partial [u_t]}{\partial s}) \lambda dw - \int_G \Omega L(u_t) d\nu.$$

The volume integral on the right side must be understood as the integral over G_1 plus the integral over G_2 . However, since u is required to have continuous third-order derivatives off $\emptyset = 0$, it follows by differentiating (2.1) that u_t satisfies $L(u) = 0$ in G_1 and in G_2 . Moreover, the discontinuities in u and its derivatives are required to be jump discontinuities. Hence the volume integral in question must be zero.

We shall now sum the results of equations (6.6) to (6.11) and add to these the analogous equations for G_2 . For the resulting volume integral on the left side we get zero by virtue of (6.1). For the resulting volume integral on the right side we use (6.12). Also we replace the (dx_n) in (6.6) to (6.11) by $\lambda \phi_t dw$ in the case of G_1 and by $-\lambda \phi_t dw$ in the case of G_2 . Instead of writing down the entire surface integral which the summation yields, let us note that the integrand contains terms in Ω , Ω_t , and Ω_{tt} . We shall treat these separately. Neglecting the λdw , we get for the coefficient of Ω_{tt}

$$[u] \left(\sum_{i,j=1}^{n-1} a^{ij} \psi_i \psi_j \phi_t - 2a^{it} \psi_i \phi_t + a^{tt} \phi_t \right).$$

If we use the fact that $\phi = t - \psi$, we may say that

$$(6.13) \quad \Omega_{tt} \rightarrow [u] \sum_{i,j=1}^n a^{ij} \phi_i \phi_j.$$

We now collect the coefficients of Ω_t . These are, neglecting λdw ,

$$\sum_{i,j=1}^{n-1} a^{ij} \frac{\partial [u]}{\partial x_j} \psi_i \phi_t + \sum_{i,j=1}^{n-1} a^{ij} [u] \psi_{ij} \phi_t + \sum_{i,j=1}^{n-1} a^{ij} \frac{\partial [u]}{\partial x_i} \psi_j \phi_t - 2 \sum_{i=1}^{n-1} a^{it} \frac{\partial [u]}{\partial x_i} \phi_t$$

$$- \sum_{k=1}^n b^k [u] \phi_k - [u_t] \sum_{i,j=1}^n a^{ij} \phi_i \phi_j .$$

Replacing derivatives of ψ by derivatives of $\phi = t - \psi$ and expanding derivatives of $[u]$, wherein it is understood that $u = u(x_1, x_2, \dots, x_{n-1}, \psi)$, we obtain

$$- \sum_{i,j=1}^{n-1} a^{ij} [u_j] \phi_i + \sum_{i,j=1}^{n-1} a^{ij} [u_t] \phi_i \phi_j - \sum_{i,j=1}^{n-1} a^{ij} [u] \phi_{ij} - \sum_{i,j=1}^{n-1} a^{ij} [u_i] \phi_j + \sum_{i,j=1}^{n-1} a^{ij} [u_t] \phi_i \phi_j$$

$$- 2 \sum_{i=1}^{n-1} a^{it} [u_i] \phi_t + 2 \sum_{i=1}^{n-1} a^{it} [u_t] \phi_i - \sum_{k=1}^n b^k [u] \phi_k - [u_t] \sum_{i,j=1}^n a^{ij} \phi_i \phi_j .$$

If we use the fact²⁰ that $\sum_{i,j=1}^n a^{ij} \phi_i \phi_j = 0$, we may replace the second and seventh

terms by $-a^{nn} [u_t] \phi_t \phi_t$ and the resulting expression agrees precisely with (5.14).

Hence we may apply the steps leading from (5.14) to (5.18) and write that

$$(6.14) \quad \Omega_t \rightarrow - [u_t] \sum_{i,j=1}^n a^{ij} \phi_i \phi_j - \bar{L}(\phi) [u] - 2 \frac{\partial [u]}{\partial s} ,$$

the first term coming from the volume integral (6.12) and the remainder from other terms in (6.6) to (6.11).

There remains now the coefficient of Ω . These terms, aside from the factors λdw and ϕ_t , are as follows:

$$(6.15) \quad \Omega \rightarrow \sum_{i,j=1}^{n-1} a^{ij} \frac{\partial^2 [u]}{\partial x_i \partial x_j} + \sum_{k=1}^{n-1} b^k \frac{\partial [u]}{\partial x_k} + c[u] + \bar{L}(\phi) [u_t] + 2 \frac{\partial [u_t]}{\partial s} .$$

The last two terms come from the volume integral in (6.12) and the rest from other terms in (6.6) to (6.11). In (6.15) it must be remembered that $[u] = [u(x_1, x_2, \dots, x_{n-1}, \psi)]$.

We may now determine how the quantities in (6.13), (6.14) and (6.15) are related. If $[u] \neq 0$, then the argument in section V based on the independence of Ω and Ω_t showed that the characteristic equation and the first transport equation hold. Then the coefficients of Ω_{tt} and Ω_t as given in (6.13) and (6.14) are 0. If $[u] = 0$ on

20. In section V we argued that the coefficients of Ω and Ω_t in the surface integral are independently zero because of the arbitrariness in Ω . Then if $[u] \neq 0$, the characteristic equation holds and so may be used here. However we shall show in a moment that even if $[u] = 0$ but $[u_t] \neq 0$ then the characteristic equation still holds.

$\phi = 0$, then certainly the coefficient of Ω_{tt} here is 0 and, since the first transport equation is 0, we may throw out this part of the coefficient of Ω_t . Again arguing from the independence of Ω and Ω_t we see first from (6.14) that if $[u_t] \neq 0$ on $\phi = 0$, the characteristic equation must hold. That is, discontinuities of u_t must lie on a surface $\phi = 0$ which satisfies the characteristic equation even if u itself is continuous on $\phi = 0$. Secondly, the coefficient of Ω must also be zero. This last fact therefore yields the second transport equation which reads

$$(6.16) \quad \bar{L}(\phi)[u_t] + 2 \frac{\partial [u_t]}{\partial s} = - \sum_{i,j=1}^{n-1} a^{ij} \frac{\partial^2 [u]}{\partial x_i \partial x_j} - \sum_{k=1}^{n-1} b^k \frac{\partial [u]}{\partial x_k} - c[u].$$

This equation must hold on $\phi = t-\psi = 0$. The operators on the right side apply to $[u] = [u(x_1, \dots, x_{n-1}, \psi)]$. It will be noted that these operators involve only spatial differentiations or differentiations inside or along $\phi = 0$. Equation (6.16) is a non-homogeneous linear, first-order ordinary differential equation in $[u_t]$ which holds along the rays²¹.

To solve equation (6.16) we note that the non-homogeneous term would be obtained from the solution of the first transport equation. This solution would be obtained as a function of s . However, equation (5.16) gives the x_i as functions of parameters α_i and s . This family of rays must be on the surface $\phi = 0$ and hence the family depends upon $n-1$ parameters and s . By inverting this system, that is, solving for the $n-1$ parameters and s , and substituting these solutions in $[u]$, we obtain this quantity as a function of the n -variables $x_1, \dots, x_n$, and then may obtain the derivatives which appear on the right side as functions of the x_i . We may then replace the x_i in the derivatives by their values in terms of s to solve the equation (6.16).

We can now see how to derive the transport equations for $[u_{tt}]$ and the higher time discontinuities. We start, by analogy with equation (6.1), with the equation

$$(6.17) \quad \int_G uM(\Omega_{tt}) dv = 0.$$

We now suppose that Ω_{tt} and all its first derivatives vanish on Γ . Again we regard the integral over G as a sum of integrals over G_1 and G_2 . Analogously to (6.2), we may state that

$$(6.18) \quad \begin{aligned} \int_{G_2} u(a^{ij}\Omega)_{ijtt} dv &= \int_{\phi} u(a^{ij}\Omega)_{ijtt} (dx_n) - \int_{G_1} u_t(a^{ij}\Omega)_{ijtt} dv, \\ &= \int_{\phi} u(a^{ij}\Omega)_{ijtt} (dx_n) - \int_{\phi} u_t(a^{ij}\Omega)_{ij} (dx_n) + \int_{G_1} u_{tt}(a^{ij}\Omega)_{ij} dv. \end{aligned}$$

21. Compare Courant-Hilbert, loc.cit., pp.362-363 and equations (12) on p.363 in particular. The quantity J there is the right hand side here. Equation (6.16) above is, apart from the method of derivation, a more general result than equation (12) there. We have the form of the transport equation for the case where the characteristic surface is any $\phi = 0$. Moreover, the relationship of (6.16) or the Courant-Hilbert equation (12) to the transport equation for $[u]$ itself is now established.

We now compare the last two terms here with the two terms on the right side of (6.2). The differences are in sign and in that u_t here replaces u there. We may treat the last two terms here as in steps (6.2) to (6.15) and obtain the various coefficients of Ω , Ω_t , and Ω_{tt} given by (6.13) to (6.15) after changes in sign and substitution of u_t for u .

The first term on the right side of (6.18) is analogous to the first term on the right side of (6.2) except that Ω_t replaces Ω . Again we may repeat the steps from (6.2) to (6.15), noting, however, only those contributions to (6.13), (6.14) and (6.15) which come from the surface integral of (6.2) and that the coefficients of Ω_{tt} , Ω_t , and Ω will now be coefficients of Ω_{ttt} , Ω_{tt} , Ω_t respectively. The following will result for the coefficients of the various derivatives of Ω .

$$(6.19) \quad \Omega_{ttt} \rightarrow [u] \sum_{i,j=1}^n a^{ij} \phi_i \phi_j$$

$$(6.20) \quad \Omega_{tt} \rightarrow - [u_t] \sum_{i,j=1}^n a^{ij} \phi_i \phi_j - \bar{L}(\phi)[u] - 2 \frac{\partial [u]}{\partial s}$$

$$(6.21) \quad \Omega_t \rightarrow + [u_{tt}] \sum_{i,j=1}^n a^{ij} \phi_i \phi_j + \bar{L}(\phi)[u_t] + 2 \frac{\partial [u_t]}{\partial s} + \sum_{i,j=1}^{n-1} a^{ij} \frac{\partial^2 [u]}{\partial x_i \partial x_j} \\ + \sum_{k=1}^{n-1} b^k \frac{\partial [u]}{\partial x_k} + c[u]$$

$$(6.22) \quad \Omega \rightarrow - \sum_{i,j=1}^{n-1} a^{ij} \frac{\partial^2 [u_t]}{\partial x_i \partial x_j} - \sum_{k=1}^{n-1} b^k \frac{\partial [u_t]}{\partial x_k} - c[u_t] - \bar{L}(\phi)[u_{tt}] - 2 \frac{\partial [u_{tt}]}{\partial s}.$$

We now may argue as we did in connection with the derivation of (6.16) from (6.13) to (6.15). If $[u] \neq 0$ on $\phi = 0$, the characteristic equation and first and second transport equations hold. Hence the coefficients of Ω_{ttt} , Ω_{tt} , and Ω_t vanish, and since Ω is arbitrary we get the third transport equation:

$$(6.23) \quad \bar{L}(\phi)[u_{tt}] + 2 \frac{\partial [u_{tt}]}{\partial s} = - \sum_{i,j=1}^{n-1} a^{ij} \frac{\partial^2 [u_t]}{\partial x_i \partial x_j} - \sum_{k=1}^{n-1} b^k \frac{\partial [u_t]}{\partial x_k} - c[u_t].$$

If $|u| = 0$, but $|u_t| \neq 0$ on $\phi = 0$, then as we saw above, the characteristic equation holds, and of course the first transport equation holds automatically. Moreover, arguing from the independence of Ω and Ω_t , we see from (6.21) that the second transport equation holds. From (6.22) we obtain (6.23). If $[u] = 0$ and $[u_t] = 0$ then the first and second transport equations hold automatically. If at the same time $[u_{tt}] \neq 0$, then (6.21) shows that the characteristic equation holds, and we have the fact that discontinuities of $[u_{tt}]$ which are not discontinuities of u or u_t must nevertheless lie on a $\phi = 0$ which satisfies the characteristic equation. Moreover, (6.22) shows that (6.23) holds in this case too.

It is apparent that we can obtain higher transport equations from our basic integral equation. Systematization of the process would make the procedure mechanical.

We may now relate the general theory of discontinuities to the problem of obtaining an asymptotic solution of our partial differential equation. The theory applies to any discontinuous solution and hence certainly applies to the pulse solution whose discontinuities furnish the coefficients of the expansion. What distinguishes this solution from the others is the initial conditions which would uniquely determine the solutions of the transport equations. We consider this point in the next section.

VII. Initial Conditions

If the transport and higher transport equations are to be used to furnish the coefficients of the asymptotic expansion, initial conditions are needed. No general prescription can be given. However, it does seem possible in several cases to state what the initial conditions should be

a) If a plane wave of the form e^{ikx} impinges on a perfectly conducting surface, e.g., the exterior of a parabolic cylinder, and one seeks the asymptotic expansion of the reflected field, then we may use the fact that the sum of the incident and reflected field is zero on the conductor and hence require that each coefficient in the asymptotic expansion of the reflected field be the negative of that of the incident field. Since e^{ikx} is the first and only term of its own asymptotic expansion, in which e^{ikx} is the phase factor and the amplitude coefficient is 1, we may require that the initial value of the first term of the reflected field be -1 and the initial value of each of the other coefficients be 0.

b) Suppose the theory applicable to a scalar problem, let us say, with a point source in a non-homogeneous medium. The scalar potential of a Hertzian dipole is $\frac{e^{ikr}}{r}$ in a homogeneous medium. This term is also its own asymptotic expansion in powers of λ . To obtain the corresponding potential in asymptotic form for a dipole source in a non-homogeneous medium we require that each coefficient of the desired expansion approach, at $r=0$, the corresponding coefficient of $\frac{e^{ikr}}{r}$. To avoid the infinities involved we note first that the analogue of r in non-homogeneous media is the $\sqrt{K}$, where K is the expansion coefficient of the reciprocal of curvature. Hence we multiply the n -th term of the desired expansion by $(\sqrt{K})^n$ and require that this term approach the n -th term of the expansion of $\frac{e^{ikr}}{r}$ multiplied by r^n . Of course the latter term will be zero except for $n=1$.

c) Initial conditions may be obtained from the boundary conditions imposed in problem I defined in section II. The pulse solution $U(x,t)$ is required by definition to satisfy the boundary condition (2.1c) where $f(t)$ is $\eta(t)$. Then $U(x,0^+) = h(x)$ and by (2.1a), $U(x,0^-) = 0$. Hence $[U(x,0)] = h(x)$. Moreover, for x on S_1 , $[U_t(x,0)] = 0$, and likewise for the higher time derivatives. If the variable t is used as the parameter along the rays, then we do have the initial values for the transport equations, at least insofar as the field which propagates along rays issuing from S_1 is concerned.

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